

DSF-97/14
hep-th/9704144

Covariant Majorana Formulation of Electrodynamics

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Abstract

We construct an explicit covariant Majorana formulation of Maxwell electromagnetism which does not make use of vector 4-potential. This allows to write a “Dirac” equation for the photon containing all the known properties of it. In particular, the spin and (intrinsic) boost matrices are derived and the helicity properties of the photon are studied.

1 Introduction.

Recently, many different experiments have revealed new electromagnetic phenomena that, although they can be well described by Maxwell electromagnetism, sound quite unusual with respect to our traditional view of electrodynamics.

An example is the observation of tunneling photons with group velocities also greater than c ; this effect does not violate Einstein causality principle and can be described in terms of Maxwell equations (see [1, 2] and references therein).

Another long discussed example is the experimental measurement of the Evans-Vigier B^3 field in non-linear optical experiments (such as in the inverse Faraday effect, etc.) [3], the B^3 field being the phase free spin field of Maxwell electromagnetism. To this regard, Evans has interestingly noted [4] that the cyclic equations for the B^3 field [3] are particularly evident in the Majorana formulation of electrodynamics.¹

Majorana's original idea [6] was that if the Maxwell theory of electromagnetism has to be viewed as the wave mechanics of the photon, then it must be possible to write Maxwell equations as a Dirac-like equation for a probability quantum wave ψ , this wave function being expressible by means of the physical $\vec{\mathcal{E}}$, $\vec{\mathcal{B}}$ fields. This can be, indeed, realized introducing the quantity

$$\vec{\psi} = \vec{\mathcal{E}} - i\vec{\mathcal{B}} \quad (1)$$

since $\vec{\psi}^* \cdot \vec{\psi} = \vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2$ is directly proportional to the probability density function for a photon². In terms of $\vec{\psi}$, the Maxwell equations in vacuum then write [7]

$$\vec{\nabla} \cdot \vec{\psi} = 0 \quad (2)$$

$$\frac{\partial \vec{\psi}}{\partial t} = i\vec{\nabla} \times \vec{\psi} \quad (3)$$

By making use of the correspondence principle

$$E \rightarrow i \frac{\partial}{\partial t} \quad (4)$$

$$\vec{p} \rightarrow -i\vec{\nabla} \quad (5)$$

these equations effectively can be cast in a Dirac-like form

$$(E - \vec{\alpha} \cdot \vec{p}) \vec{\psi} = 0 \quad (6)$$

¹Although more indirectly, also in the theoretical description of tunneling photons the employment of the writing of Maxwell equations as a Dirac-like equation for the photon seems quite useful; this can be seen confronting [2] and [5]. The author is grateful to A. Ranfagni for bringing to his attention these two references.

²If we have a beam of n equal photons each of them with energy ϵ (given by the Planck relation), since $\frac{1}{2}(\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2)$ is the energy density of the electromagnetic field, then $\frac{1}{n\epsilon} \frac{1}{2}(\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2) dS dt$ gives the probability that each photon has to be detected in the area dS in the time dt . The generalization to photons of different energies (i.e. of different frequencies) is obtained with the aid of the superposition principle.

with the transversality condition

$$\vec{p} \cdot \vec{\psi} = 0 \quad (7)$$

where the 3x3 hermitian matrices $(\alpha_i)_{lm} = i \epsilon_{ilm}$

$$\alpha^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad \alpha^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \alpha^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (8)$$

satisfying

$$[\alpha_i, \alpha_j] = -i \epsilon_{ijk} \alpha_k \quad (9)$$

have been introduced.

Note that the probabilistic interpretation is indeed possible given the “continuity equation” (Poynting theorem)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \quad (10)$$

where

$$\rho = \frac{1}{2} \vec{\psi}^* \cdot \vec{\psi} \quad \vec{j} = -\frac{1}{2} \psi^* \vec{\alpha} \psi \quad (11)$$

are respectively the energy and momentum density of the electromagnetic field.

The fascination of Majorana formulation of electrodynamics lies mainly in the fact that it deals with directly observable quantities, such as the electric and the magnetic fields, without the occurrence of electromagnetic potentials. On the other side, we know that Maxwell equations are Lorentz invariant, but this is not manifest in the cited formulation. Actually, the lack of a covariant Majorana formulation is due to the fact that in the known covariant formulation of electrodynamics it is of fundamental importance the introduction of the vector 4-potential A_μ .

The main goal of this paper is to show how to obtain a genuine covariant Majorana formulation, without the use of the 4-potential. In the next section we develop the appropriate formalism to this end, while in section 3 the covariant Majorana-Maxwell equations are derived and their properties are pointed out. In section 4 we construct the Majorana hamiltonian and deal with spin and (intrinsic) boost matrices, and with the photon helicity. Finally, the conclusions.

2 Covariant Kinematics of the Electromagnetic Field.

Covariant electromagnetism is described by the field strength tensors $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$. The basic property of these tensors is their antisymmetric behaviour under the exchange of their indices μ and ν ; this can be expressed saying that for any 4-vector u_μ the relations

$$u_\mu u_\nu F^{\mu\nu} = 0 \quad u_\mu u_\nu \tilde{F}^{\mu\nu} = 0 \quad (12)$$

must hold. Without loss of generality, we can assume the 4-vector u_μ to be unitary, so that

$$u_\mu u^\mu = 1 \quad (13)$$

Let us now introduce the following “auxiliary fields” (u_μ -dependent)

$$E^\mu = u_\nu F^{\mu\nu} \quad B^\mu = u_\nu \tilde{F}^{\mu\nu} \quad (14)$$

in terms of which the property (12) writes

$$u_\mu E^\mu = 0 \quad u_\mu B^\mu = 0 \quad (15)$$

i.e. the 4-vectors E_μ , B_μ are both orthogonal to u_μ . We are now able to express the tensors $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ by means of the auxiliary fields in the following way:

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu + \epsilon_{\mu\nu\alpha\beta} B^\alpha u^\beta \quad (16)$$

$$\tilde{F}_{\mu\nu} = u_\mu B_\nu - u_\nu B_\mu - \epsilon_{\mu\nu\alpha\beta} E^\alpha u^\beta \quad (17)$$

We stress that the fields E_μ , B_μ depends on u_μ in such a manner that $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are independent on u_μ itself. We see, in fact, that the auxiliary fields E_μ , B_μ are nothing that covariant linear u_μ -combinations of the physical electric and magnetic field

$$\mathcal{E}^i = F^{0i} \quad \mathcal{B}^i = \tilde{F}^{0i} \quad (18)$$

Namely, using (13), (15), (16), (17), the relations between the auxiliary fields and the physical fields are the following:

$$E_0 = \vec{u} \cdot \vec{\mathcal{E}} \quad \vec{E} = u_0 \vec{\mathcal{E}} + \vec{u} \times \vec{\mathcal{B}} \quad (19)$$

$$B_0 = \vec{u} \cdot \vec{\mathcal{B}} \quad \vec{B} = u_0 \vec{\mathcal{B}} - \vec{u} \times \vec{\mathcal{E}} \quad (20)$$

For a given u_μ , the fields E_μ , B_μ are univocally determined by the electric and magnetic fields. Note that in the “isotropic case”, in which

$$u_\mu = (1, \vec{0}) \quad (21)$$

we obtain $E^\mu = (0, \vec{\mathcal{E}})$, $B^\mu = (0, \vec{\mathcal{B}})$.

It is interesting to observe the analogy between the relations (19), (20) and those for the transformations of the electric and magnetic field from an inertial reference frame to another moving with relative velocity \vec{v} with $|\vec{v}| \ll 1$ [8]:

$$\vec{\mathcal{E}}' \simeq \vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}} \quad \vec{\mathcal{B}}' \simeq \vec{\mathcal{B}} - \vec{v} \times \vec{\mathcal{E}} \quad (22)$$

This suggestive analogy allows to interpret the 3-vector part of the auxiliary fields as the Lorentz transformed electric and magnetic fields and \vec{u} as a velocity.

3 Covariant Majorana form of Maxwell equations.

Armed with the formalism developed in the previous section, we are now able to write Maxwell equations according to Majorana's point of view. Here we are interested in illustrating the method, so that we will consider only non-interacting photons; Maxwell equations in vacuum are then

$$\partial_\mu F^{\mu\nu} = 0 \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \quad (23)$$

Before continuing, let us note that equations (23) imply

$$\partial_\mu E^\mu = 0 \quad \partial_\mu B_\mu = 0 \quad (24)$$

Now, assuming u_μ independent on space-time coordinates, when we substitute (16), (17) in (23) we have

$$u^\mu \partial_\mu E^\nu + \epsilon^{\mu\nu\alpha\beta} \partial_\mu B_\alpha u_\beta = 0 \quad (25)$$

$$u^\mu \partial_\mu B^\nu - \epsilon^{\mu\nu\alpha\beta} \partial_\mu E_\alpha u_\beta = 0 \quad (26)$$

Introducing the complex field

$$\psi_\mu = E_\mu - i B_\mu \quad (27)$$

satisfying

$$\partial_\mu \psi^\mu = 0 \quad (28)$$

equations (23) rewrite as

$$\partial_\mu (u^\mu \psi^\nu + i \epsilon^{\mu\nu\alpha\beta} \psi_\alpha u_\beta) = 0 \quad (29)$$

Then, defining the following four 4x4 matrices

$$(\gamma^\mu)^\alpha_\beta = u^\mu g^\alpha_\beta + i \epsilon^{\mu\alpha}_{\beta\gamma} u^\gamma \quad (30)$$

the equations of motion of the electromagnetic field in vacuum acquire the form (using (4), (5))

$$(\gamma^\mu)^\alpha_\beta p_\mu \psi^\beta = 0 \quad (31)$$

This is just the “Dirac equation” for free photons.

3.1 Properties of γ -matrices.

The explicit form of the four γ -matrices is the following:

$$\gamma^0 = \begin{pmatrix} u^0 & 0 & 0 & 0 \\ 0 & u^0 & i u^3 & -i u^2 \\ 0 & -i u^3 & u^0 & i u^1 \\ 0 & i u^2 & -i u^1 & u^0 \end{pmatrix} \quad (32)$$

$$\gamma^1 = \begin{pmatrix} u^1 & 0 & -i u^3 & i u^2 \\ 0 & u^1 & 0 & 0 \\ -i u^3 & 0 & u^1 & i u^0 \\ i u^2 & 0 & -i u^1 & u^1 \end{pmatrix} \quad (33)$$

$$\gamma^2 = \begin{pmatrix} u^2 & i u^3 & 0 & -i u^1 \\ i u^3 & u^2 & 0 & -i u^0 \\ 0 & 0 & u^2 & 0 \\ -i u^1 & i u^0 & 0 & u^2 \end{pmatrix} \quad (34)$$

$$\gamma^3 = \begin{pmatrix} u^3 & -i u^2 & i u^1 & 0 \\ -i u^2 & u^3 & i u^0 & 0 \\ i u^1 & -i u^0 & u^3 & 0 \\ 0 & 0 & 0 & u^3 \end{pmatrix} \quad (35)$$

These are hermitian matrices

$$\left((\gamma^\mu)^\alpha_\beta \right)^* = (\gamma^\mu)^\alpha_\beta \quad (36)$$

satisfying the relations

$$u_\mu \gamma^\mu = 1 \quad (37)$$

$$Tr \gamma^\mu = 4 u^\mu \quad (38)$$

The algebra of γ -matrices is defined by

$$[\gamma^\mu, \gamma^\nu] = \frac{i}{4} \epsilon^{\mu\nu\alpha\beta} (Tr \gamma_\alpha) \gamma_\beta \quad (39)$$

Explicitly, in terms of u_μ , their commutator can be expressed as follows:

$$[\gamma^\mu, \gamma^\nu]^\alpha_\beta = u^\alpha \left(u^\mu g^\nu_\beta - u^\nu g^\mu_\beta \right) + (g^{\mu\alpha} u^\nu - g^{\nu\alpha} u^\mu) u_\beta - (g^{\mu\alpha} g^\nu_\beta - g^{\nu\alpha} g^\mu_\beta) \quad (40)$$

3.2 The “isotropic case” $u_\mu = (1, \vec{0})$.

In the special case in which $u_\mu = (1, \vec{0})$ the γ -matrices reduce to

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad (41)$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (42)$$

so that $(\gamma^i)^{lm} = i \epsilon^{ilm}$ coincide with the α -matrices in (8). The equations of motion then reproduce exactly the Majorana-Maxwell equations (2), (3)

$$\vec{p} \cdot \vec{\psi} = 0 \quad (43)$$

$$E \vec{\psi} + i \vec{p} \times \vec{\psi} = 0 \quad (44)$$

3.3 Properties of the equations of motion.

From (31), (30) we have:

$$(u \cdot p) \psi^\mu = i \epsilon^\mu_{\alpha\beta\gamma} p^\alpha \psi^\beta u^\gamma \quad (45)$$

For $u \cdot p \neq 0$ (in the limit (21) this selects solutions with $E \neq 0$), multiplying (45) by u_μ and summing we obtain

$$u_\mu \psi^\mu = 0 \quad (46)$$

In an analogous way, multiplying by p_μ we get also

$$p_\mu \psi^\mu = 0 \quad (47)$$

From these, we directly see that the equations of motions (31) contains the orthogonality conditions (15) and (28). For further applications, it is useful to deduce also the “adjoint equation” of (31); by means of the hermiticity condition (36), we then have

$$\psi_\alpha^* (\gamma^\mu)^\alpha_\beta p_\mu = 0 \quad (48)$$

(where we understand that $p_\mu = i \overleftarrow{\partial}_\mu$ acts on ψ^*). The comparison of (48) and (31) makes evident the fact that photons coincide with antiphotons [7].

3.4 Probability current: the continuity equation.

Multiplying (31) by ψ^* and (48) by ψ and summing, we easily obtain the equation

$$\partial_\mu J^\mu = 0 \quad (49)$$

where the current J^μ is given by ³

$$J^\mu = -\frac{1}{2} \psi^* \gamma^\mu \psi \quad (50)$$

and satisfies

$$u_\mu J^\mu = -\frac{1}{2} \psi^* \cdot \psi \quad (51)$$

The “probability density” J^0 in (50) is a well-defined positive quantity (as one can easily check), and in the limit (21) eq. (49) reduces to the Poynting theorem (10).

³Note that the factor $\frac{1}{2}$ can be eliminated by a redefinition of the fields ψ and ψ^* , while the minus sign can be absorbed in the definition of γ -matrices.

However, in the general case (with an arbitrary u_μ) it is more correct to consider directly the energy-momentum tensor

$$T^{\mu\nu} = -F^{\mu\sigma} F^\nu_\sigma + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (52)$$

whose components $T^{0\mu}$ are proportional to the probability current density, as explained in section 1 ($T^{00} = \frac{1}{2}(\mathcal{E}^2 + \mathcal{B}^2)$, $T^{0i} = \mathcal{E} \times \mathcal{B}$). Substituting (16) and (30) in (52) we have the following expression for the tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = -\frac{1}{4}(\psi^* \{\gamma^\mu, \gamma^\nu\} \psi + (\psi^{*\mu} \psi^\nu + \psi^{*\nu} \psi^\mu)) \quad (53)$$

where $\{\dots, \dots\}$ denotes the anticommutator. Using (40), $T^{\mu\nu}$ can be cast in the asymmetric form

$$T^{\mu\nu} = -\frac{1}{2} \{\psi^* \gamma^\mu \gamma^\nu \psi + \psi^{*\mu} \psi^\nu\} \quad (54)$$

Another useful writing of $T^{\mu\nu}$ in terms of u_μ is the following:

$$\begin{aligned} T^{\mu\nu} = & \left(\frac{1}{2} g^{\mu\nu} - u^\mu u^\nu \right) \psi^* \cdot \psi - \frac{i}{2} \left(u^\mu \epsilon^\nu_{\alpha\beta\gamma} + u^\nu \epsilon^\mu_{\alpha\beta\gamma} \right) \psi^{*\alpha} \psi^\beta u^\gamma \\ & - \frac{1}{2} (\psi^{*\mu} \psi^\nu + \psi^{*\nu} \psi^\mu) \end{aligned} \quad (55)$$

Using, then, the equations of motion one can prove that $T^{\mu\nu}$ is conserved:

$$\partial_\nu T^{\mu\nu} = 0 \quad (56)$$

The general expression for the probability current density is, finally, given by $T^{0\mu}$.

The fact that we have found two conserved currents, J^μ and $T^{0\mu}$, which coincide only in the limit (21), induces to ask to ourselves why this happen. In reality, the conservation of J^μ and $T^{0\mu}$ are two expressions of the same physical principle, that is the energy-momentum conservation (or, in other words, the probability conservation). In fact, one can easily prove that

$$T^{\mu\nu} = u^\mu J^\nu + u^\nu J^\mu - (u \cdot J) g^{\mu\nu} + j^{\mu\nu} \quad (57)$$

where

$$j^{\mu\nu} = -\frac{1}{2} \psi^* \tau^{\mu\nu} \psi \quad (58)$$

$$(\tau^{\mu\nu})^\alpha_\beta = g^{\mu\alpha} g^\nu_\beta + g^{\nu\alpha} g^\mu_\beta \quad (59)$$

is a particular symmetric tensor satisfying

$$u_\mu j^{\mu\nu} = 0 \quad (60)$$

The current J^μ is then given by

$$J^\mu = u_\nu T^{\mu\nu} \quad (61)$$

from which it is clear that the conservation of J^μ follows from that of $T^{\mu\nu}$, as stated earlier.

A very peculiar role is played by the current $j^{0\mu}$; as we shall see in the next section, this is a Lorentz boost term, whose presence is not evident in the non-covariant formulation of Majorana-Maxwell electromagnetism. Obviously, this “boost current” is not a conserved one,

$$\partial_\mu j^{0\mu} = \partial^0 (u \cdot J) - (u \cdot \partial) J^0 \quad (62)$$

and it is interesting to observe, from (60), that it does not contribute to the current J^μ .

4 Hamiltonian form. Spin and intrinsic boost.

From the covariant equation of motion (31) one can deduce a useful Lorentz non-invariant hamiltonian formulation left-multiplying (31) by γ_0^{-1} . The equations of motion can then be cast in the form $H\psi = E\psi$ with

$$H = \vec{a} \cdot \vec{p} \quad \vec{a} = \gamma_0^{-1} \vec{\gamma} \quad (63)$$

The explicit form of the matrix γ_0^{-1} is given by

$$\left(\gamma_0^{-1}\right)^\alpha{}_\beta = u_0 g^\alpha{}_\beta - \left(g^{\alpha 0} u_\beta + u^\alpha g^0{}_\beta\right) + \frac{u^\alpha u_\beta}{u_0} + \frac{g^{\alpha 0} g^0{}_\beta}{u_0} - i \epsilon^{0\alpha}{}_{\beta\gamma} u^\gamma \quad (64)$$

and the following relations hold true:

$$u_0 \left(\gamma_0^{-1} + \gamma_0\right) = u_0 \left(\gamma_0^* - \gamma_0\right) + \left(\gamma_0^2 + 1\right) \quad (65)$$

$$\left(\gamma_0^{-1}\right)^\dagger = \gamma_0^{-1} \quad (66)$$

the latter saying that γ_0^{-1} is hermitian, as the other γ -matrices. It is then easy to see that, in the form (63), the matrices \vec{a} are not hermitian, in general.

However, this is just an accident since, using (46) and the $\mu = 0$ equation of (45), we can recast the equations of motion in the form $H\psi = E\psi$ with, now,

$$H = (\vec{\alpha} + \vec{\tau}) \cdot \vec{p} \quad (67)$$

where the matrices

$$\begin{aligned} \alpha^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} & \alpha^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} & \alpha^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \tau^1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \tau^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \tau^3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

are involved. Note that

$$\left(\alpha^i\right)^\alpha{}_\beta = -i\epsilon^{0i\alpha}{}_\beta \quad (68)$$

are the generalization to 4 dimensions of the $\vec{\alpha}$ -matrices in (8), while

$$\left(\tau^i\right)^\alpha{}_\beta = g^{0\alpha}g^i{}_\beta + g^{i\alpha}g^0{}_\beta \quad (69)$$

coincide with the matrices τ^{0i} defined in (59).

It is somewhat interesting to observe that the hamiltonian in (67) is explicitly independent on u_μ (and holds true for arbitrary u_μ), the equations of motion $H\psi = E\psi$ being themselves explicitly independent on u_μ :

$$E\psi_0 = \vec{p} \cdot \vec{\psi} \quad (70)$$

$$E\vec{\psi} = \vec{p}\psi_0 - i\vec{p} \times \vec{\psi} \quad (71)$$

However, not all the solutions of (70), (71) are allowed, since in this form these equations does not contain the orthogonality condition (46). This is a fundamental condition, expressing the antisymmetric behaviour of $F_{\mu\nu}$, $\tilde{F}_{\mu\nu}$; so together with (70), (71) also the equation (46) must be considered, and the allowed solutions of $H\psi = E\psi$ are those satisfying (46).

Obviously, in the limit (21) we recover the Majorana-Maxwell equations (43), (44).

Let us now turn to the matrices (68) and (69). These verifies the following commutation rules:

$$[\alpha^i, \alpha^j] = -i\epsilon^{ijk}\alpha^k \quad (72)$$

$$[\alpha^i, \tau^j] = -i\epsilon^{ijk}\tau^k \quad (73)$$

$$[\tau^i, \tau^j] = -i\epsilon^{ijk}\alpha^k \quad (74)$$

and we recognize in these the Lorentz algebra.

By requiring that the total angular momentum of the photon $\vec{J} = \vec{L} + \vec{S}$ is conserved (i.e. $[H, \vec{J}] = 0$) we can then identify the spin matrices with

$$\vec{S} = -\vec{\alpha} \quad (75)$$

In an analogous way, we can further identify the intrinsic part of Lorentz boost matrices with

$$\vec{K}_s = i\vec{\tau} \quad (76)$$

as we have already anticipated.

With the introduction of the spin matrices, we can now rewrite the equations of motion in the form

$$\frac{\vec{S} \cdot \vec{p}}{E} \psi = -\psi + \frac{\vec{\tau} \cdot \vec{p}}{E} \psi \quad (77)$$

where in the L.H.S. we recognize the helicity operator. With a direct calculus, using (47), we see that

$$\frac{\vec{\tau}_\nu^\mu \cdot \vec{p}}{E} \psi^\nu = \frac{p^\mu}{E^2} (\vec{p} \cdot \vec{\psi}) \quad (78)$$

so that the boost term in the equations of motion is proportional to $\vec{p} \cdot \vec{\psi}$. If $\vec{p} \cdot \vec{\psi} = 0$ the boost term vanishes, and then *only* in this case the photon is in an helicity eigenstate (with $\lambda = \pm 1$ eigenvalues). But, from (47), $\vec{p} \cdot \vec{\psi} = 0$ means $\psi_0 = 0$ and this, from (46), implies that $u_\mu = (1, \vec{0})$.

We then conclude that the physical transversality of the photon implies:

- the photon is in an helicity eigenstate (with $\lambda = \pm 1$ eigenvalues);
- a Lorentz boost term in the equations of motion is forbidden.

Inversely:

- if the photon has also longitudinal degrees of freedom, then it is not in an helicity eigenstate (as happens for the Dirac equation too);
- longitudinal degrees of freedom (*if present* [3]) are described by a Lorentz boost term in the equations of motion.

5 Conclusions.

In this paper we have given a version of the Majorana formulation of Maxwell electromagnetism which is explicitly invariant under the Lorentz group. The main advantage of the Majorana formulation is that it does not require the use of electromagnetic potentials, thus avoiding non-physical degrees of freedom. This has been here achieved also in the explicit Lorentz invariant version with the introduction of “auxiliary fields”, linear (covariant) combinations of the electric and magnetic fields built using the basic antisymmetric properties of the electromagnetic field strenghts $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$.

As also in the non-covariant version [6, 7], the Majorana formulation of electromagnetism is strongly based on the fact that there are 3 spatial and 1 temporal dimensions, since only in 3+1 dimensions the dual tensor of $F_{\mu\nu}$ is a two indices tensor as $F_{\mu\nu}$.

Here we have shown not only that Maxwell equations can be written as a Dirac-like equation for the photon, but also that the interpretative structure of the Dirac equation can be maintained for the photon as well. In fact, interpreting ψ^* as the adjoint field of ψ (see eq. (48)), the probability current for the photon has the same form as in the Dirac case (cfr. eq. (50)), provided the new γ -matrices are defined as in (30).

The “transversality” property of the photon (47) (together with (46)) is contained in the Dirac-like equation (31) and it is thus not necessary a further constraint, as in the non-covariant formulation (see eq. (7)).

Finally, the spin matrices, but also the (intrinsic) boost ones, have been derived in section 4 using the hamiltonian form of the Dirac-like equation (31), and the helicity properties of the photon has been recovered as well.

Acknowledgements

This paper takes its origin by a fruitful mailing correspondence with Prof. E. Recami: the author is indebted with him.

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